Eur. Phys. J. B **20**, 153–156 (2001)

THE EUROPEAN PHYSICAL JOURNAL B
EDP Sciences

Dynamical pinning and non-Hermitian mode transmutation in the Burgers equation

H.C. Fogedby[a]

Institute of Physics and Astronomy[b], University of Aarhus, 8000, Aarhus C, Denmark
and
NORDITA, Blegdamsvej 17, 2100, Copenhagen Ø, Denmark

Received 8 November 2000

Abstract. We discuss the mode spectrum in both the deterministic and noisy Burgers equations in one dimension. Similar to recent investigations of vortex depinning in superconductors, the spectrum is given by a non-Hermitian eigenvalue problem which is related to a 'quantum' problem by a complex gauge transformation. The soliton profile in the Burgers equation serves as a complex gauge field engendering a *mode transmutation* of diffusive modes into propagating modes and giving rise to a *dynamical pinning* of localized modes about the solitons.

PACS. 05.10.Gg Stochastic analysis methods (Fokker-Planck, Langevin, etc.) – 05.45.-a Nonlinear dynamics and nonlinear dynamical systems – 64.60.Ht Dynamic critical phenomena – 05.45.Yv Solitons

The noisy Burgers equation and the related Kardar-Parisi-Zhang (KPZ) equation provide a continuum description of an intrinsically nonequilibrium noise-driven system. As such they delimit an interesting class of systems far from equilibrium. Specifically, the equations apply to the growth of an interface either due to a random drive or subject to random environments.

In the case of one spatial dimension, which is our concern here, the Burgers equation for the local slope $u = \nabla h$ has the form [1]

$$\left(\frac{\partial}{\partial t} - \lambda u \nabla\right) u = \nu \nabla^2 u + \nabla \eta. \tag{1}$$

The related KPZ equation for the height h [2] is $\partial h/\partial t = \nu\nabla^2 h + (\lambda/2)(\nabla h)^2 + \eta$. The damping constant ν characterizes the linear diffusive term. The coupling strength λ controls the nonlinear growth or mode coupling term. The noise η is spatially short-ranged Gaussian white noise correlated according to $\langle\eta(xt)\eta(00)\rangle = \Delta\delta(x)\delta(t)$ and characterized by the strength Δ.

The stochastic equation (1) and its KPZ version have been studied intensively in particular in recent years and much insight concerning the pattern formation and scaling properties engendered by these equations has been gained on the basis of i) field theoretical approaches [3], ii) mapping to directed polymers [4], and iii) mapping to the asymmetric exclusion model [5].

In recent works [6] we advanced a Martin-Siggia-Rose based canonical phase space approach to the noisy Burgers equation (1). This method applies in the weak noise limit $\Delta \to 0$ and replaces the stochastic Burgers equation (1) by two coupled deterministic mean field equations

$$\left(\frac{\partial}{\partial t} - \lambda u \nabla\right) u = \nu \nabla^2 u - \nabla^2 p\ , \tag{2}$$

$$\left(\frac{\partial}{\partial t} - \lambda u \nabla\right) p = -\nu \nabla^2 p\ , \tag{3}$$

for the slope field u and a canonically conjugate *noise field* p, characterizing the noise η.

The appearance of the noise field p is an intrinsic feature of the Martin-Siggia-Rose functional approach, where averaging over the noise η, implementing the Burgers equation (1) as a delta function constraint, gives rise to an extra field p in addition to u. Another way is to regard the functional Fokker-Planck equation associated with the Burgers equation as an effective "Schrödinger equation". The noise field p is then the canonically conjugate variable to the slope u; this "doubling of variables" is a general feature of the deterministic path integral formulation of the stochastic problem and was also encountered in the context of mapping an interface onto a spin model [7].

The field equations derive from a principle of least action with Hamiltonian density $\mathcal{H} = p(\nu\nabla^2 u + \lambda u\nabla u - (1/2)\nabla^2 p)$ and determine orbits in a canonical phase space spanned by u and p. Moreover, the action associated with a finite time orbit from u' to u, $S = \int_{0,u'}^{t,u} \mathrm{d}t\mathrm{d}x(p\partial u/\partial t - \mathcal{H})$, provides direct access to the transition probability $P(u' \to u, t) \propto \exp[-S/\Delta]$ and associated correlations; an important aspect which was pursued in [6].

[a] e-mail: `fogedby@ifa.au.dk`
[b] *Permanent address*

On the 'zero noise' manifold for $p = 0$ the field equation (2) reduces to the noiseless Burgers equation [8]

$$\left(\frac{\partial}{\partial t} - \lambda u \nabla\right) u = \nu \nabla^2 u \ , \tag{4}$$

which, as a nonlinear evolution equation exhibiting transient pattern formation, has been used to model 'turbulence' and for example galaxy formation [9], see also [10].

In this paper we discuss two new features associated with the pattern formation in the Burgers equation in both the noiseless case (4) and the noisy case in terms of (2) and (3). We focus on the interplay between localized nonlinear soliton modes and superposed linear modes and show i) the soliton-induced *mode transmutation* of diffusive modes into propagating modes and ii) the *dynamical pinning* of linear modes about the solitons. Details will appear elsewhere.

It is a feature of the nonlinear growth terms that the field equations (2) and (3) admit nonlinear localized soliton solutions, in the static case of the kink-like form

$$u_{\rm s}^{\mu} = \mu u \tanh[k_{\rm s} x] \ , \quad k_{\rm s} = \frac{\lambda u}{2\nu}, \quad \mu = \pm 1 \ . \tag{5}$$

The index μ labels the *right hand* soliton for $\mu = 1$ with $p_{\rm s} = 0$, also a solution of the damped noiseless Burgers equation for $\eta = 0$; and the noise-excited *left hand* soliton for $\mu = -1$ with $p_{\rm s} = 2\nu u_{\rm s}$, a solution of the growing (unstable) noiseless Burgers equation for $\nu \to -\nu$. The amplitude-dependent wavenumber $k_{\rm s}$ sets the inverse soliton length scale. Noting that the field equations (2) and (3) are invariant under the slope-dependent Galilean transformation

$$x \to x - \lambda u_0 t \ , \qquad u \to u + u_0 \ , \tag{6}$$

propagating solitons are generated by the Galilean boost (6). Denoting the right and left boundary values by u_+ and u_-, respectively, the propagating velocity is given by the soliton condition

$$u_+ + u_- = -2v/\lambda \ . \tag{7}$$

It follows from the quasi-particle representation advanced in [6], see also [7], that a general interface slope profile $u = u_{\rm s} + \delta u$ at a particular instant can be represented by a dilute gas of solitons amplitude-matched according to (5) with superposed linear modes δu. For a configuration consisting of n solitons we then have

$$u_{\rm s} = \frac{2\nu}{\lambda} \sum_{p=1}^{n} k_p \tanh |k_p|(x - v_p t - x_p) \ , \tag{8}$$

where we have introduced the mean amplitude of the pth soliton $k_p = (\lambda/4\nu)(u_{p+1} - u_p)$, u_{p+1} and u_p are the boundary values. The velocity of the p-th soliton is $v_p = -(\lambda/2)(u_{p+1} + u_p)$, x_p is the center of mass, and we are assuming vanishing boundary conditions $u_1 = u_{n+1} = 0$. Note that the configuration (8) is only valid at times in between soliton collisions; the interface changes dynamically subject to the conservation of energy, momentum, and total area $\int {\rm d}x \ u$ under the slope profile. The number of solitons, however, is not conserved, see [6,7]. In Figure 1. we have depicted an n-soliton configuration.

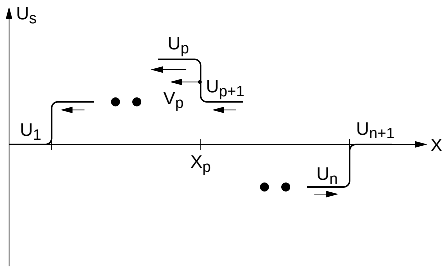

Fig. 1. We depict an n-soliton slope configuration of a growing interface. The p-th soliton moves with velocity $v_p = -(\lambda/2)(u_{p+1} + u_p)$, has boundary value u_+ and u_-, and is centered at x_p. The arrows on the horizontal inter-soliton segments indicate the propagation of linear modes.

In order to discuss the linear mode spectrum about the soliton configuration $u_{\rm s}$ it is convenient to introduce the shifted noise field φ

$$p = \nu(u - \varphi) \ . \tag{9}$$

The field equations (2) and (3) then assume the symmetrical form, also discussed in [7],

$$\left(\frac{\partial}{\partial t} - \lambda u \nabla\right) u = \nu \nabla^2 \varphi \ , \tag{10}$$

$$\left(\frac{\partial}{\partial t} - \lambda u \nabla\right) \varphi = \nu \nabla^2 u \ . \tag{11}$$

In the linear Edwards-Wilkinson case [11] for $\lambda = 0$ the field equations readily support a diffusive mode spectrum of extended growing and decaying modes, $u \pm \varphi \propto \exp(\mp \nu k^2 t) \exp(ikx)$, *i.e.*, $u = [A \exp(-\nu k^2 t) + B \exp(\nu k^2 t)] \exp(ikx)$, consistent with the phase space interpretation discussed in [6], see also [7].

Expanding about $u_{\rm s}$ and the associated noise field $\varphi_{\rm s}$ ($\varphi_{\rm s}^{\mu} = \mu u_{\rm s}^{\mu}$ for $\mu = \pm 1$),

$$\varphi_{\rm s} = \frac{2\nu}{\lambda} \sum_{p=1}^{n} |k_p| \tanh |k_p|(x - v_p t - x_p) \ , \tag{12}$$

$u = u_{\rm s} + \delta u$ and $\varphi = \varphi_{\rm s} + \delta\varphi$, the superposed linear mode spectrum is governed by the coupled non-Hermitian eigenvalue equations

$$\left(\frac{\partial}{\partial t} - \lambda u_{\rm s} \nabla\right) \delta u = \nu \nabla^2 \delta\varphi + \lambda(\nabla u_{\rm s}) \delta u \ , \tag{13}$$

$$\left(\frac{\partial}{\partial t} - \lambda u_{\rm s} \nabla\right) \delta\varphi = \nu \nabla^2 \delta u + \lambda(\nabla \varphi_{\rm s}) \delta u \ . \tag{14}$$

In the inter-soliton matching regions of constant slope field $\nabla u_{\rm s} = \nabla \varphi_s = 0$ and the equations (13) and (14) decouple as in the Edwards-Wilkinson case. Setting $u_{\rm s} = u$ and searching for solutions of the form $\delta u, \delta\varphi \propto \exp(-E_k t) \exp(ikx)$ we obtain $\delta u \pm \delta\varphi \propto \exp(-E_k^{\pm} t) \exp(ikx)$, *i.e.*, $\delta u = [A \exp(-E_k^+ t) + B \exp(-E_k^- t)] \exp(ikx)$, where the complex spectrum characteristic of a non-Hermitian eigenvalue problem is given by

$$E_k^{\pm} = \pm\nu k^2 - \mathrm{i}\lambda u k \ . \tag{15}$$

Introducing the phase velocity $v = \lambda u$ the δu mode corresponds to a propagating wave with both a growing and decaying component

$$\delta u \propto (A\mathrm{e}^{-\nu k^2 t} + B\mathrm{e}^{\nu k^2 t})\mathrm{e}^{\mathrm{i}k(x+vt)} \ . \tag{16}$$

The presence of the nonlinear soliton profile thus gives rise to a *mode transmutation* in the sense that the diffusive mode in the Edwards-Wilkinson case is transmuted to a propagating mode (16) in the Burgers case. As indicated in Figure 1 the linear mode propagates to the left for $u > 0$ and to the right for $u < 0$. We note in particular that for a static *right hand* soliton $u_\pm = \pm u$ and the mode propagates towards the soliton center which thus acts like a 'sink'; for a static *left hand* soliton the situation is reversed, the mode propagates away from the soliton which in this case plays the role of a 'source'.

In the soliton region the slope field varies over a scale given by k_s^{-1} and we must address the equations (13) and (14). Introducing the auxiliary variables

$$\delta X^{\pm} = \delta u \pm \delta\varphi \ , \tag{17}$$

they take the form

$$-\frac{\partial \delta X^{\pm}}{\partial t} = \pm D\left(\pm\frac{\lambda}{2\nu}u_\mathrm{s}\right)\delta X^{\pm} - \frac{\lambda}{2}(\nabla u_\mathrm{s} \pm \nabla\varphi_\mathrm{s})\delta X^{\mp} \ , \tag{18}$$

where $D(\pm\lambda u_s/2\nu)$ is the 'gauged' Schrödinger operator

$$D\left(\pm\frac{\lambda}{2\nu}u_\mathrm{s}\right) = -\nu(\nabla \pm \frac{\lambda}{2\nu}u_\mathrm{s})^2 + \frac{\lambda^2}{4\nu}u_\mathrm{s}^2 - \frac{\lambda}{2}\nabla\varphi_s \ , \tag{19}$$

for the motion of a particle in the potential $(\lambda/4\nu)u_\mathrm{s}^2 - (\lambda/2)\nabla\varphi_\mathrm{s}$ subject to a gauge field $(\lambda/2\nu)u_\mathrm{s}$ given by u_s.

In the regions of constant slope field $\nabla u_\mathrm{s} = \nabla\varphi_s = 0$, $u_\mathrm{s} = u$, $D(\pm\lambda u_\mathrm{s}/2\nu) \to -\nu(\nabla \pm \lambda u/2\nu)^2 + (\lambda/4\nu)u^2$ and searching for solutions of the form $\delta X^{\pm} \propto \exp(-E_k t)\exp(\mathrm{i}kx)$ we recover the spectrum (15). In the soliton region $\nabla\varphi_\mathrm{s}^\mu = \mu\nabla u_\mathrm{s}^\mu$, $\mu = \pm 1$, and one of the equations (18) decouples driving the other equation parametrically.

In order to pursue the analysis of (18) we note that the gauge field $\lambda u_\mathrm{s}/2\nu$ in the Schrödinger operator can be absorbed by means of the gauge or Cole-Hopf [12] transformation

$$U = \exp\left[-\frac{\lambda}{2\nu}\int \mathrm{d}x \ u_\mathrm{s}\right] \ . \tag{20}$$

Using the relation $D(\pm\lambda u_\mathrm{s}/2\nu) = U^{\pm 1}D(0)U^{\mp 1}$ we obtain the Hermitian eigenvalue equations

$$-\frac{\partial \delta X^{\pm}}{\partial t} = \pm U^{\pm 1}D(0)U^{\mp 1}\delta X^{\pm} - \frac{\lambda}{2}(\nabla u_\mathrm{s} \pm \nabla\varphi_\mathrm{s})\delta X^{\mp} \ , \tag{21}$$

which are readily analyzed in terms of the spectrum of $D(0)$ discussed in [10].

The exponent or generator in the gauge transformation (20) samples the area under the slope profile u_s up to the point x. For $x \to \infty$, $U \to \exp[-\lambda M/2\nu]$, where $M = \int \mathrm{d}x \ u_\mathrm{s}$ is the total area; according to (1) or (2) M is a conserved quantity. In terms of the height field h, $u = \nabla h$, $M = h(+L) - h(-L)$ is equal to the height offset across a system of size L, *i.e.*, a conserved quantity under growth. Inserting the soliton profile u_s (8) the transformation U factorizes in contributions from local solitons, *i.e.*,

$$U = \prod_{p=1}^{n} U_p^{\mathrm{sign}k_p} \ , \ U_p = \cosh^{-1} k_p(x - v_p t - x_p) \ . \tag{22}$$

Focusing on a particular soliton contribution to the interface with boundary values u_+ and u_- and for convenience located at $x_p = 0$ the analysis is most easily organized by first performing a Galilean transformation (6) to a local rest frame by shifting the slope field by $(u_+ + u_-)/2$, corresponding to the velocity given by (7). The static soliton is then given by (5), $u_\pm = \pm u$, and

$$D(0) = -\nu\nabla^2 + \nu k_\mathrm{s}^2\left[1 - 2/\cosh^2 k_\mathrm{s}x\right] \ , \tag{23}$$

describing the motion of a particle in the attractive potential $-2\nu k_\mathrm{s}^2/\cosh^2 k_\mathrm{s}x$ whose spectrum is known.

Denoting the eigenvalue problem $D(0)\Psi_n = \Omega_n\Psi_n$ the spectrum of $D(0)$ is composed of a zero-energy $\Omega_0 = 0$ localized state $\Psi_0 \propto \cosh^{-1} k_\mathrm{s}x$, yielding the soliton translation mode lifting the broken translational symmetry, and a band $\Psi_k \propto \exp(\mathrm{i}k_\mathrm{s}x)s_k(x)$ of extended phase-shifted scattering modes with energy

$$\Omega_k = \nu(k^2 + k_\mathrm{s}^2) \ . \tag{24}$$

$s_k(x) = (k + \mathrm{i}k_\mathrm{s}\tanh k_\mathrm{s}x)/(k - \mathrm{i}k_\mathrm{s})$ is a modulation of the plane wave state; for $x \to \infty$, $s_k(x) \to \exp(\mathrm{i}\delta_k)$, where δ_k is the phase shift of the wave.

Inserting $\nabla\varphi_\mathrm{s}^\mu = \mu\nabla u_s^\mu$ the fluctuations

$$\delta\tilde{X}^{\pm} = U_p^{\mp\mu}\delta X^{\pm} \ , \ U_p = \cosh^{-1} k_\mathrm{s}x \ , \tag{25}$$

then satisfy the Hermitian eigenvalue equations

$$-\frac{\partial \delta\tilde{X}^{\pm}}{\partial t} = \pm D(0)\delta\tilde{X}^{\pm} - \nu k_\mathrm{s}^2(\mu \pm 1)\delta\tilde{X}^{\mp} \ , \tag{26}$$

which decouple and are analyzed by expanding $\delta\tilde{X}^{\pm}$ on the eigenstates Ψ_n. For the *right hand* soliton for $\mu = +1$ and focusing on the plane wave component, ignoring phase shift effects, we obtain in particular the fluctuations

$$\delta X^{+} = (A\mathrm{e}^{-\Omega_k t} + B\mathrm{e}^{\Omega_k t})\mathrm{e}^{\mathrm{i}kx}\cosh^{-1} k_\mathrm{s}x \ , \tag{27}$$

$$\delta X^{-} = B\frac{\Omega_k}{\nu k_\mathrm{s}^2}\mathrm{e}^{\Omega_k t}\mathrm{e}^{\mathrm{i}kx}\cosh k_\mathrm{s}x. \tag{28}$$

For real k the modes $\delta X^{\pm}$ are diffusive and the spectrum $\Omega_k = \nu(k^2 + k_\mathrm{s}^2)$ exhibits a gap νk_s^2 proportional to the soliton amplitude squared. Moreover, the gauge transformation U_p gives rise to a spatial modulation of the plane wave form which allows us to extend the spectrum by an analytical continuation in the wavenumber k. In particular by setting $k \to k \mp \mathrm{i}k_\mathrm{s}$ for $\delta X^{\pm}$ and noting that $\delta X^{\pm}$ decouple for $x \gg k_\mathrm{s}^{-1}$ we have $\Omega_k \to \nu k^2 \mp 2\mathrm{i}\nu k k_\mathrm{s}$ and $\exp(\mathrm{i}kx)\cosh^{\mp 1} k_\mathrm{s}x \to$ const. and we achieve a matching to the extended propagating modes in the inter-soliton regions. A similar analysis applies to the *left hand* soliton for $\mu = -1$ and the gauge transformation (22) allows for a complete analysis of the linear fluctuation spectrum about the multi-soliton configuration u_s.

The phenomena of mode transmutation has also been noted by Schütz [13] in the case of the asymmetric exclusion model, a lattice version of the noisy Burgers equation, in the context of analyzing the shocks, corresponding to the solitons in the present context.

The last issue we wish to address is the fluctuation spectrum in the noiseless case, extending the analysis in [10]. In this case we only have the *right hand* soliton for $\mu = +1$ and the soliton and associated fluctuations lie on the zero-noise manifold for $p = 0$. There is no coupling to the 'noisy' modes *i.e.*, $u = \varphi$, $\delta u = \delta\varphi = \delta X^+/2$, and $\delta X^- = 0$, and the fluctuations are given by (27) for $B = 0$ (ignoring phase shift effects)

$$\delta u = \delta X^+/2 \propto \mathrm{e}^{-\Omega_k t}\mathrm{e}^{\mathrm{i}kx}\cosh^{-1} k_s x \ . \qquad (29)$$

It is an essential feature of the non-Hermitian eigenvalue problem (18) characterizing noisy nonequilibrium growth, and in the noiseless case of transient growth yielding (29), that the real spectrum (24) can be extended into the complex eigenvalue plane. This is due to the envelope $\cosh^{-1} k_s x$ which gives rise to a spatial fall off. Setting $k \to k + \mathrm{i}\kappa$, $\Omega_k \to E_{k,\kappa}$, where

$$E_{k,\kappa} = \nu(k^2 + k_s^2 - \kappa^2) + 2\mathrm{i}\nu k\kappa \ . \qquad (30)$$

For $|\kappa| < k_s$ we have a band of localized fluctuations *dynamically pinned* to the soliton. The modes are exponentially damped with a damping constant given by the real part of $E_{k,\kappa}$, $\mathrm{Re}E_{k,\kappa} = \nu(k_s^2 - \kappa^2)$. The imaginary part of $E_{k,\kappa}$, $\mathrm{Im}E_{k,\kappa} = 2\nu k\kappa$, combined with the phase $\mathrm{i}kx$ yields a propagating wave with phase velocity $2\nu\kappa$, finally the spatial range of the mode is given by $(k_s - \kappa)^{-1}$. For $\kappa = 0$ the spectrum is real, the phase velocity vanishes, the range is k_s^{-1}, and the localized mode is symmetric and purely diffusive with a gap k_s^2. For $\kappa = k_s$, the borderline case, the fluctuations are extended in space and purely propagating with a gapless spectrum νk^2. For intermediate κ values the modes are propagating with localized envelopes. In Figure 2 we have depicted the complex eigenvalue spectrum.

In this paper we have analyzed two aspects of the interplay between linear superposed modes and nonlinear soliton excitations in the noiseless and noisy Burgers equations describing transient and stationary nonequilibrium growth, respectively. Both aspects are intimately related to the non-Hermitian character of the eigenvalue problem. The first aspect is a linear mode transmutation where the diffusive non-propagating modes in the linear Edwards-Wilkinson case owing to the solitons are transmuted to propagating extended modes in the nonlinear Burgers case. The second aspect is a dynamical pinning of a band of localized modes to the solitons. We finally note that similar aspects are also encountered in recent work on the transverse Meissner effect and flux pinning in superconductors [14]. Here the uniform gauge field is given by the transverse magnetic field whereas in our case the nonlinear soliton profile provides the nonuniform gauge field.

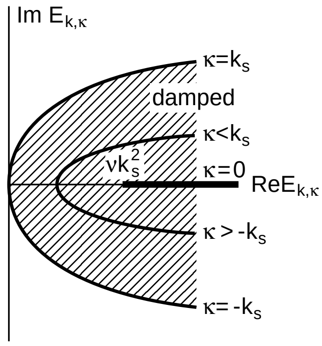

Fig. 2. We depict the the complex eigenvalue spectrum for the damped modes in the noiseless Burgers equation. On the boundaries $\kappa = \pm k_s$ the modes are extended and propagating. The shaded area indicates propagating modes with localized envelopes. For $\kappa = 0$ the mode is symmetrical and purely diffusive.

Discussions with A. Svane, K. Mølmer, J. Hertz, B. Derrida, M. Lässig, J. Krug and G. Schütz are gratefully acknowledged.

References

1. D. Forster, D.R. Nelson, M.J. Stephen, Phys. Rev. Lett. **36**, 867 (1976); D. Forster, D.R. Nelson, M.J. Stephen, Phys. Rev. A **16**, 732 (1977).
2. M. Kardar, G. Parisi, Y.C. Zhang, Phys. Rev. Lett. **56**, 889 (1986); E. Medina, T. Hwa, M. Kardar, Y.C. Zhang, Phys. Rev. A **39**, 3053 (1989).
3. T. Hwa, E. Frey, Phys. Rev. A **44**, R7873 (1991); U.C. Täuber, E. Frey, Phys. Rev. E **51**, 6319 (1995); E. Frey, U.C. Täuber, T. Hwa, Phys. Rev. E **53**, 4424 (1996); E. Frey, U.C. Täuber, H.K. Janssen, Europhys. Lett. **47**, 14 (1999); M. Lässig, Nucl. Phys. B **448**, 559 (1995); M. Lässig, Phys. Rev. Lett. **80**, 2366 (1998).
4. T. Halpin-Healy, Y.C. Zhang, Phys. Rep. **254**, 215 (1995).
5. B. Derrida, J.L. Lebowitz, Phys. Rev. Lett. **80**, 209 (1998).
6. H.C. Fogedby, Phys. Rev. E **59**, 5065 (1999); H.C. Fogedby, Phys. Rev. E **60**, 4950 (1999).
7. H.C. Fogedby, A.B. Eriksson, L.V. Mikheev, Phys. Rev. Lett. **75**, 1883 (1995); H.C. Fogedby, Phys. Rev. E **57**, 4943 (1998); H.C. Fogedby, Phys. Rev. Lett. **80**, 1126 (1998).
8. J.M.Burgers, Proc. Roy. Neth. Acad. Soc. **32**, 414, 643, 818 (1929).
9. P. Saffman, *Topics in Nonlinear Physics*, edited by N.J. Zabusky (Springer, New York, 1968); Y. Zeldovitch, Astron. Astrophys. **8**, 84 (1972).
10. H.C. Fogedby, Phys. Rev. E **57**, 2331 (1998).
11. S.F. Edwards, D.R. Wilkinson, Proc. Roy. Soc. London A **381**, 17 (1982).
12. J.D. Cole, Quart. Appl. Math. **9**, 22 (1951); E. Hopf, Comm. Pure Appl. Math. **3**, 201 (1950).
13. G.M. Schütz, in *Phase Transitions and Critical Phenomena*, Vol. 19 edited by C. Domb, J. Lebowitz (Academic Press, London 2000).
14. N. Hatano, D.R. Nelson, Phys. Rev. Lett. **77**, 570 (1996); N. Hatano, D.R. Nelson, Phys. Rev. E **56**, 8651 (1997).